The Conjugate (Dual) of an Operator.

Definition: — Let N be a normed linear space and let T be an operator on N, that is let T be a continuous linear transformation of N into itself. Define a linear transformation T^* of N^* into itself as follows: if $f \in N^*$, then $T^*(f)$ is given by

$$[T^*(f)](x) = f(T(x)) \quad \text{——— (A)}$$

We call T^* the conjugate (or adjoint) of T.

Theorem: — Let T be an operator on a normed linear space N, then its conjugate T^* defined by

$$T^* : N^* \to N^* : T^*(f) = f \circ T$$

and $[T^*(f)](x) = f(T(x))$ — (1) for all $f \in N^*$ and all $x \in N$, is an operator on N^* and the mapping

$$\phi : B(N) \to B(N^*) : \phi(T) = T^* \ \forall\, T \in B(N)$$

is an isometric isomorphism of $B(N)$ into $B(N^*)$ which reverses products and preserves the identity transformation.

Proof. — T^* is clearly linear on N^*. Hence in order that T^* may be an operator on N^*, we must show that it is bounded. We have

$$\|T^*\| = \sup\{\|T^*(f)\| : \|f\| \le 1\}.$$

$$= \sup\{|T^*(f)(x)| : \|f\| \le 1, \|x\| \le 1\}.$$

$$= \sup\{|f\,T(x)| : \|f\| \le 1, \|x\| \le 1\} \text{ by } 1.$$

[illegible] $\leq \sup\{\|f\|\,\|T\|\,\|x\| : \|f\| \leq 1, \|x\| \leq 1\}$

$$\leq \|T\| \quad \text{———} \quad (2)$$

Since T is bounded, (2) shows that T^* is bounded and so T^* is an opeator on N^*.

Again, for each non-zero vector x in N there exists a functional $f \in N^*$ such that $\|f\| = 1$

$$\text{and } f(T(x)) = \|T(x)\| \quad \text{———} \quad (3)$$

And therefore

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\}$$

$$= \sup\left\{\frac{|f(T(x))|}{\|x\|} : \|f\| = 1, x \neq 0\right\} \text{ by (3)}$$

$$= \sup\left\{\frac{|[T^*(f)](x)|}{\|x\|} : \|f\| = 1, x \neq 0\right\} \text{ by (1)}$$

$$= \sup\left\{\frac{\|T^*(f)\|\,\|x\|}{\|x\|} : \|f\| = 1, x \neq 0\right\}.$$

$$= \sup\{\|T^*(f)\| : \|f\| = 1\} = \|T^*\| \quad \text{———} \quad (4)$$

From (2) and (4)

$$\|T^*\| = \|T\|. \quad \text{———} \quad (5)$$

We now show that the mapping

$$\Phi : B(N) \to B(N^*) : \Phi(T) = T^* \ \forall\ T \in B(N) \quad \text{———} \quad (6)$$

is an Isometric Isomorphism that is, we show that Φ is a one-one linear transformation such that

$$\|\Phi(T)\| = \|T\|$$

Now, $\|\Phi(T)\| = \|T^*\|$ by (6) $= \|T\|$ by (5).

Φ is linear.

Let T, U be arbitrary elements of $B(N)$ and α, β any scalars.

Then $\Phi(\alpha T + \beta U) = (\alpha T + \beta U)^*$ by (6)

But $[(\alpha T+\beta U)^*(f)](x) = f((\alpha T+\beta U)(x))$ by (1)

$= f(\alpha T(x)+\beta U(x))$

$= \alpha f(T(x)) + \beta f(U(x))$ [$\because f$ is linear]

$= \alpha[T^*(f)](x) + \beta[U^*(f)](x)$ by (1)

$= [\alpha[T^*(f)] + \beta[U^*(f)]](x)$

$\therefore (\alpha T+\beta U)^*(f) = \alpha[T^*(f)] + \beta[U^*(f)]$

$= (\alpha T^* + \beta U^*)(f)$

So that $(\alpha T+\beta U)^*(f) = \alpha T^* + \beta U^*$ ——— (7)

$\therefore \phi(\alpha T+\beta U) = (\alpha T+\beta U)^* = \alpha T^* + \beta U^*$

$= \alpha\phi(T) + \beta\phi(U)$.

Thus ϕ is linear.

ϕ is one-one.

$\phi(T) = \phi(U) \Rightarrow T^* = U^*$

$\Rightarrow \|T^* - U^*\| = 0$

$\Rightarrow \|(T-U)^*\| = 0$ [Take $\alpha=1$, $\beta=-1$ in (7)]

$\Rightarrow \|T-U\| = 0$ by (5)

$\Rightarrow T = U$.

Hence ϕ is shown to be an isometric isomorphism. What remains to show is that x reverses products and preserves the identity transformation, we have

$[(TU)^*(f)](x) = f((TU)(x))$ — by (1)

$= f(T(U(x)))$

$= [T^*(f)](U(x))$ by (1) since $U(x) \in N$.

Name : Date / / Page No.

$$= [U^*(T^*(f))](x) \quad \text{by } (1)$$

$$\text{Hence } (TU)^* = U^* T^* \quad \text{———} \quad (8)$$

So that $\phi(TU) = (TU)^* = U^* T^*$

Thus ϕ reverses products.

Finally if I denotes the identity operator on H, then

$$[I^*(f)](x) = f(I(x)) \quad \text{— by } (1)$$

$$= f(x) = (If)(x)$$

$$\text{Hence } I^* = I$$

So that $\phi(I) = I^* = I$

And thus ϕ preserves the identity transformation

— Proved.

Anjani Kumar Singh.